Applications of Numerical Homotopy Continuation to Mechanism Design

Mark Plecnik
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Nonlinear Algebra in Applications
Motivation

Inventing machines through computation...
Central Design Element

Linkages:

Planar

Spherical

Spatial

Images courtesy UC Irvine Robotics & Automation Laboratory
Typical Problem Statement

Trace a plane curve:

**One approach:**
Break curve into discrete points

How to size a linkage?

http://www.partsw.com
A Simplified History

The simplest linkage:

A crank...
...can go through 3 points

Result date: Unknown
### A Simplified History

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>No. of Points</th>
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Synthesis Objectives

**Function generation**: set of input angles and output angles;

**Motion generation**: set of positions and orientations of a workpiece;

**Path generation**: set of points along a trajectory in the workpiece.
Synthesis Procedures

Specify task

Choose linkage type

(A separate design challenge) →

Formulate high degree equations

Optimization methods

Direct solution

Graphical

Human intuition

(More common approach) →

(Our approach)
Literature Review


Function Generator

Coordinate input crank with output crank

Task *(specified)*

\((0, 0), (\phi_1, \psi_1), (\phi_2, \psi_2), (\phi_3, \psi_3), (\phi_4, \psi_4), (\phi_5, \psi_5), (\phi_6, \psi_6), (\phi_7, \psi_7), (\phi_8, \psi_8), (\phi_9, \psi_9), (\phi_{10}, \psi_{10})\)

\[ Q = e^{i\phi} \quad S = e^{i\psi} \]

Joint coordinates *(unknowns)*

\[ \begin{array}{cccccccc}
A & B & C & D & F & G & H \\
\end{array} \]

Rotation operators *(extra unknowns)*

\[ R = e^{i\rho} \quad T = e^{i\theta} \quad U = e^{i\mu} \]

Loop equations *(constraints)*

\[ A + Q_j (C - A) + R_j (G - C) = B + S_j (D - B) + T_j (G - D), \]
\[ A + Q_j (C - A) + R_j (H - C) = B + S_j (F - B) + U_j (H - F), \quad j = 1, \ldots, N - 1 \]
Synthesis Equations

• Loop equations:
  \( A + Q_j(C - A) + R_j(G - C) = B + S_j(D - B) + T_j(G - D) \),
  \( A + Q_j(C - A) + R_j(H - C) = B + S_j(F - B) + U_j(H - F) \),

\( j = 1, \ldots, N - 1 \)

• Conjugate loop equations:
  \( \overline{A} + \overline{Q}_j(\overline{C} - \overline{A}) + \overline{R}_j(\overline{G} - \overline{C}) = \overline{B} + \overline{S}_j(\overline{D} - \overline{B}) + \overline{T}_j(\overline{G} - \overline{D}) \),
  \( \overline{A} + \overline{Q}_j(\overline{C} - \overline{A}) + \overline{R}_j(\overline{H} - \overline{C}) = \overline{B} + \overline{S}_j(\overline{F} - \overline{B}) + \overline{U}_j(\overline{H} - \overline{F}) \),

\( j = 1, \ldots, N - 1 \)

• Rotation operators:
  \( R_j \overline{R}_j = 1, \quad T_j \overline{T}_j = 1, \quad U_j \overline{U}_j = 1, \quad j = 1, \ldots, N - 1 \)

• Unknowns:
  \( \{ C, \overline{C}, D, \overline{D}, F, \overline{F}, G, \overline{G} \} \),
  \( \{ R_j, \overline{R}_j, T_j, \overline{T}_j, U_j, \overline{U}_j \}, \quad j = 1, \ldots, N - 1 \)

System square for \( N = 11 \),
70 eqns and unknowns, degree = \( 1.18 \times 10^{21} \)
Algebraic Reduction

\[
\begin{align*}
A + Q_j(C - A) + R_j(G - C) &= B + S_j(D - B) + T_j(G - D), \\
\overline{A} + \overline{Q}_j(C - \overline{A}) + \overline{R}_j(G - \overline{C}) &= \overline{B} + \overline{S}_j(D - \overline{B}) + \overline{T}_j(G - \overline{D}), \\
A + Q_j(C - A) + R_j(H - C) &= B + S_j(F - B) + U_j(H - F), \\
\overline{A} + \overline{Q}_j(C - \overline{A}) + \overline{R}_j(H - \overline{C}) &= \overline{B} + \overline{S}_j(F - \overline{B}) + \overline{U}_j(H - \overline{F}).
\end{align*}
\]

These unknowns are eliminated:

\[R_j, \overline{R}_j, \ T_j, \overline{T}_j, \ U_j, \overline{U}_j, \ j = 1, \ldots, N - 1\]

\[T_j \overline{T}_j = 1\]

\[U_j \overline{U}_j = 1\]

\[R_j \overline{R}_j = 1\]

\[
\begin{align*}
\begin{bmatrix}
ab_j & \bar{a}b_j \\
\bar{c}d_j & \bar{c}d_j
\end{bmatrix}
\begin{bmatrix}
R_j \\
\overline{R}_j
\end{bmatrix} &= \begin{bmatrix}
\bar{f}f - a\bar{a} - b_j\bar{b}_j \\
g\bar{g} - c\bar{c} - d_j\bar{d}_j
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
(ab_j(g\bar{g} - c\bar{c} - d_j\bar{d}_j) - c\bar{d}_j(ff - a\bar{a} - b_j\bar{b}_j))(ab_j(g\bar{g} - c\bar{c} - d_j\bar{d}_j) - c\bar{d}_j(ff - a\bar{a} - b_j\bar{b}_j)) + (ab_j\bar{c}d_j + \bar{a}b_j\bar{c}d_j)^2 &= 0
\end{align*}
\]

\[
\begin{align*}
a &= G - C, \quad b_j = A - B + Q_j(C - A) - S_j(D - B), \quad f = G - D, \\
c &= H - C, \quad d_j = A - B + Q_j(C - A) - S_j(F - B), \quad g = H - F
\end{align*}
\]

10 synthesis equations
in 10 unknowns:

\[
\begin{align*}
\{C, \overline{C}, D, \overline{D}, F, \overline{F}, G, \overline{G}, H, \overline{H}\}
\end{align*}
\]
Degree of the Synthesis Equations

Synthesis equations:
\[
(a\overline{b}_j(gg - c\overline{c} - d_j\overline{d}_j) - c\overline{d}_j(ff - a\overline{a} - b_j\overline{b}_j))(a\overline{b}_j(gg - c\overline{c} - d_j\overline{d}_j) - c\overline{d}_j(ff - a\overline{a} - b_j\overline{b}_j))+ (a\overline{b}_j\overline{c}d_j + a\overline{b}_j\overline{c}\overline{d}_j)^2 = 0
\]
\[
j = 1, \ldots, 10
\]

• Goal: To find all of the solutions \(\langle C, \overline{C}, D, \overline{D}, F, \overline{F}, G, \overline{G}, H, \overline{H}\rangle\) of the synthesis equations
• Each polynomial is degree 8
• How many roots?
  – Using Bezout’s Theorem:
    \[8^{10} = 1.07 \times 10^9\]
  – Using a multihomogeneous grouping:
    \[264,241,152\]
    This is the lowest bound we can compute.
    Uses sparse monomial structure.

• Solution method: **Polynomial Homotopy Continuation**
Polynomial Homotopy Continuation

Goal:

Regeneration homotopy: more sophisticated approach
Types of Solutions

- Polynomial homotopy attempts to find ALL of the roots of a system, including:
  - Roots at infinity
  - Finite roots
  - Nonsingular roots
  - Singular roots

- Target system solved with regeneration homotopy
  - Used the Bertini Homotopy Software
  - 24,822,328 paths tracked
  - 1,521,037 finite, nonsingular solutions found
  - 311 hrs on 256×2.2GHz

The majority of paths track to these. Limited by multihomogeneous homotopy. Discarded quickly by regeneration. Handled efficiently with projective coordinates.

This is what we desire. In this example, less than 1% of 264,241,152 roots track to these.

Discarded quickly by regeneration.
Parameter Homotopy

The General Strategy for Solving Families of Polynomial Systems

1. Find all solutions for a numerically general system by any means possible
   - Regeneration homotopy
   - Multihomogeneous homotopy
   - Non-homotopy methods

   **Computationally expensive:**
   311 hours for a single solve
   Regen tracked 24,822,328 paths
   Found 1,521,037 solutions

2. Use the results from step 1 as start points for a homotopy that solves a specific system
   - Avoids endpoints at infinity

   **Computationally efficient:**
   2 hours per solve
   Tracked 1,521,037 paths

Once a complete solution to a system is found, we can find the solutions to similar systems fast!
Stephenson III Function Generation

• Stephenson III function generation
  – Degree: 55,050,240 for 11 positions
  – Size of general solution set: 834,441
  – Initial computation: 40 hrs on 512×2.6GHz (multihomogeneous homotopy)
  – Proceeding computations: 50 min on 64×2.2GHz (parameter homotopy)

• Design of torque cancelling linkages
  – By placing a linear torsion spring on one end, a function generator can be synthesized to create a specified torque or stiffness profile
Stroke Rehabilitation Application

- Applications for torque cancelling include stroke

From measurements in stroke survivors’ wrists
Results

Biomimetic Wing Motion – Joint Angles of the Black-billed magpie

Biomimetic Human Walking Gait – Planar Joint Angles of Hip, Knee, and Ankle
Constrained RR Method

1. Begin by specifying an RR chain

2. Select a set of 11 points to move the RR chain through

3. Inverse kinematics gives a coordinated joint angle function \((\nu_j, \zeta_j), j=0,\ldots,10\)

4. Solve for 11 point Stephenson II function generators

5. Attach function generators to the RR chain

Path generation is inverted to function generation

The resulting six-bar traces through the 11 points

Linkages still need to be verified
Example

<table>
<thead>
<tr>
<th>$j$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-5.160</td>
<td>-83.957</td>
</tr>
<tr>
<td>1</td>
<td>8.346</td>
<td>-84.026</td>
</tr>
<tr>
<td>2</td>
<td>21.993</td>
<td>-83.632</td>
</tr>
<tr>
<td>3</td>
<td>32.259</td>
<td>-82.128</td>
</tr>
<tr>
<td>4</td>
<td>33.018</td>
<td>-79.911</td>
</tr>
<tr>
<td>5</td>
<td>16.497</td>
<td>-73.889</td>
</tr>
<tr>
<td>6</td>
<td>-6.363</td>
<td>-62.120</td>
</tr>
<tr>
<td>7</td>
<td>-28.276</td>
<td>-74.865</td>
</tr>
<tr>
<td>8</td>
<td>-33.406</td>
<td>-80.964</td>
</tr>
<tr>
<td>9</td>
<td>-27.733</td>
<td>-83.440</td>
</tr>
<tr>
<td>10</td>
<td>-17.440</td>
<td>-84.032</td>
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Stephenson Path Generators

- Goal: Find dimensions of Stephenson linkages so that they move a trace point through 11 points
- Formulated as the synthesis of an RR chain constrain by a Stephenson function generator
- Solve inverse kinematics of RR chain to find joint angles
- Solve for function generators that constrain those joint angles
Design Exploration
Exploration of other gaits

SIII  SII  SII  SI
Prototyping a robot

- A leg design was selected and manufactured as a flexure linkage
- Lasercut polypropylene, each leg $\frac{1}{4}'' \times \frac{1}{4}''$
- Robot length 30 cm

Pantograph linkages replaces belts
The Design Approach

Required behaviors

Design exploration

Kinematic tuning

Final design

Define Requirements

Required Behaviors
1. Traces a straight line
2. Long stroke
3. Input pivot near line-of-action
4. Compact dimensions
5. Input link rotates over large range
6. Low mech. adv. at top of stroke
7. Constant ground reaction force
8. Angular momentum balanced
The Design Approach

- Required behaviors
- Design Exploration
- Kinematic tuning
- Final design

Generate An Atlas of Designs

[Diagram showing various designs]
The Design Approach

1. Required behaviors
2. Design exploration
3. Kinematic tuning
4. Final design

Iterative Design Optimization

- Iteration I
- Iteration II
- Iteration III
- Iteration IV
- Iteration V
- Iteration VI
- Iteration VII
- Iteration VIII
The Design Approach

- Required behaviors
- Design exploration
- Kinematic tuning
- Final design
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Stephenson II Timed Curve

Task

\[(0, P_0), (\phi_1, P_1), (\phi_2, P_2), (\phi_3, P_3),
(\phi_4, P_4), (\phi_5, P_5), (\phi_6, P_6), (\phi_7, P_7)\]

Coordinate input crank with output point
Stephenson II Timed Curve

Joint coordinates

Rotation operators

\[ Q = e^{i\phi} \quad R = e^{i\rho} \quad S = e^{i\psi} \]

\[ T = e^{i\theta} \quad U = e^{i\mu} \]

Loop equations

\[ A + Q_j(C - A) + R_j(H - C) + U_j(P_0 - H) = P_j \]

\[ B + S_j(F - B) + U_j(P_0 - F) = P_j \]

\[ A + Q_j(C - A) + R_j(G - C) - B - S_j(D - B) - T_j(G - D) = 0 \]
Stephenson II Timed Curve

Loop equations

\[ A + Q_j(C - A) + R_j(H - C) + U_j(P_0 - H) = P_j \]
\[ \bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{R}_j(\bar{H} - \bar{C}) + \bar{U}_j(\bar{P}_0 - \bar{H}) = \bar{P}_j \]
\[ B + S_j(F - B) + U_j(P_0 - F) = P_j \]
\[ \bar{B} + \bar{S}_j(\bar{F} - \bar{B}) + \bar{U}_j(\bar{P}_0 - \bar{F}) = \bar{P}_j \]
\[ A + Q_j(C - A) + R_j(G - C) - B - S_j(D - B) - T_j(G - D) = 0 \]
\[ \bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{R}_j(\bar{G} - \bar{C}) - \bar{B} - \bar{S}_j(\bar{D} - \bar{B}) - \bar{T}_j(\bar{G} - \bar{D}) = 0 \]

Extra substitutions

\[ a = A\bar{H} \quad d = \frac{D - B}{F - B} \]
\[ b = B\bar{F} \quad g = \frac{G - C}{H - C} \]
\[ c = (C - A)\bar{H} \quad k = g(P_0 - H) - d(P_0 - F) \]

Unit rotations

\[ R_j\bar{R}_j = 1 \]
\[ S_j\bar{S}_j = 1 \]
\[ T_j\bar{T}_j = 1 \]
\[ U_j\bar{U}_j = 1 \]

Several substitutions
Stephenson II Timed Curve

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<tr>
<td>$\beta_j + \tilde{\beta}_j - P_j\tilde{P}_j - P_j\tilde{P}_j = 0$</td>
</tr>
<tr>
<td>$\xi_j + \tilde{\xi}_j - P_j\tilde{P}_j - P_j\tilde{P}_j = 0$</td>
</tr>
<tr>
<td>$U_jk\tilde{\zeta}_j + \tilde{U}_j\tilde{\zeta}_j - \zeta_j\tilde{\xi}_j - k\tilde{k} + (g(H - C) + C - d(P_0 - F) - B)(\tilde{g}(\tilde{H} - \tilde{C}) + \tilde{C} - \tilde{d}(\tilde{F} - \tilde{B}) - \tilde{B}) = 0$</td>
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<th>Variables</th>
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<td>total degree $= 2^7 \times 2^7 \times 4^7 \times 2^8 \times 2^7 = 8,796,093,022,208$</td>
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</table>

| Spoiler Alert! Approx 1,500,000 finite roots |

<table>
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<th>Intermediate expressions</th>
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<td>$\beta_j = U_j(P_0(\tilde{P}_j - \tilde{A} - \tilde{Q}_j(\tilde{C} - \tilde{A})) - \tilde{P}_jH + \tilde{a} + \tilde{Q}_j\tilde{c}) + Q_j(C - A)(\tilde{P}_j - \tilde{A}) + A(\tilde{P}_j - \tilde{C} - \tilde{A}) + H(\tilde{P}_0 - \tilde{C})$</td>
</tr>
<tr>
<td>$\xi_j = U_j(P_0(\tilde{P}_j - \tilde{B}) - \tilde{P}_jF + \tilde{b}) + P_j\tilde{B} + P_0\tilde{F} - b$</td>
</tr>
<tr>
<td>$\zeta_j = A - B + Q_j(C - A) + g(P_j - A - Q_j(C - A)) - d(P_j - B)$</td>
</tr>
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</table>
Why the discrepancy?

8,796,093,022,208

1,500,000
Sparse System

Start system
\[(a_1 x + a_2 y + 1)(a_3 x + a_4 y + 1)(a_5 x + a_6 y + 1) = 0\]
\[(a_7 x + a_8 y + 1)(a_9 x + a_{10} y + 1)(a_{11} x + a_{12} y + 1) = 0\]

No. of roots: 9
Monomials: \(\{x^3, y^3, x^2 y, xy^2, x^2, y^2, xy, x, y, 1\}\)

Expanded form:
\[b_1 x^3 + b_2 y^3 + b_3 x^2 y + b_4 xy^2 + b_5 x^2 + b_6 y^2 + b_7 xy + b_8 x + b_9 y + 1 = 0\]
\[b_{10} x^3 + b_{11} y^3 + b_{12} x^2 y + b_{13} xy^2 + b_{14} x^2 + b_{15} y^2 + b_{16} xy + b_{17} x + b_{18} y + 1 = 0\]

Target system
\[c_1 x^3 + c_2 xy + c_3 y + 1 = 0\]
\[c_4 x^3 + c_5 xy + c_6 y + 1 = 0\]

No. of roots: 4
Monomials: \(\{x^3, xy, y, 1\}\)

** Start**
- \(b_1\) \rightarrow \(c_1\)
- \(b_2\) \rightarrow \(0\)
- \(b_3\) \rightarrow \(0\)
- \(b_4\) \rightarrow \(0\)
- \(b_5\) \rightarrow \(0\)
- \(b_6\) \rightarrow \(0\)
- \(b_7\) \rightarrow \(c_2\)
- \(b_8\) \rightarrow \(0\)
- \(b_9\) \rightarrow \(c_3\)

**Target**

**Homotopy**

* **a & c** coefficients are generic complex numbers
### Sparse System

**Start system**

\[(a_1 x + a_2 y + 1)(a_3 x + a_4 y + 1)(a_5 x + a_6 y + 1) = 0\]
\[(a_7 x + a_8 y + 1)(a_9 x + a_{10} y + 1)(a_{11} x + a_{12} y + 1) = 0\]

No. of roots: 9  
Monomials: \{x^3, y^3, x^2 y, xy^2, x^2, y^2, xy, x, y, 1\}

A start system with monomials that match the target would be nice!

**Target system**

\[c_1 x^3 + c_2 xy + c_3 y + 1 = 0\]
\[c_4 x^3 + c_5 xy + c_6 y + 1 = 0\]

No. of roots: 4  
Monomials: \{x^3, xy, y, 1\}

---

**Recall Stephenson II example...**

**Start system**

No. of roots: 8,796,093,022,208

**Target system**

No. of roots: 1,500,000
Random Startpoints

A randomly generated mechanism...

Its movement: Loop equations
Construct a start system with exactly the right monomials

Its dimensions: $A \ B \ C \ D \ F \ G \ H$
Provide a single solution to start system
Random Startpoints

Target System

Root 1
Root 2
Root 3
Root 4
Root 5
Root 6
Root 7
Root 8

Start system Startpoint
Finite Root Generation

Begin

Target system

Target roots

Add

Discard

Has root been collected?

Yes

One root of target system

Parameter homotopy

Generate random mechanism

Start root

Randomly select mech. configs. as tasks positions

Start system

No

Have sufficient roots been collected?

End

Yes

Has root been collected?

No
Collecting Coupons

- The process of accumulating roots through FRG is analogous to randomly picking coupons out of a box.
- There are 6 unique different colored coupons in the box.

**How many picks until all coupons have been collected once?**

**Probability of picking a new color:** 50%

**15 ± 6.2**

- Expected no. of picks
- Standard deviation

<table>
<thead>
<tr>
<th>Color</th>
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<tr>
<td>Red</td>
</tr>
<tr>
<td>Orange</td>
</tr>
<tr>
<td>Yellow</td>
</tr>
<tr>
<td>Green</td>
</tr>
<tr>
<td>Blue</td>
</tr>
<tr>
<td>Violet</td>
</tr>
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</table>
FRG Root Collection

Expected no. of trials to obtain \( n \) of \( N \) roots

\[ T_n = N(H_N - H_{N-n}) \]

Harmonic numbers

\[ H_N = \sum_{k=1}^{N} \frac{1}{k} \]
FRG Estimation

Coupon collector model

\[ T_n = N(H_N - H_{N-n}) \]

Expected no. of trials \quad Total no. of roots

Approximate coupon collector model

\[ T_n \approx N \ln \left( \frac{N}{N - n} \right) \]

Estimation equation

Percentage of roots collected \quad \hat{n} = \frac{n}{N}

New root success rate \quad \alpha = \frac{n}{T_n}

\[ \alpha = -\frac{\hat{n}}{\ln(1 - \hat{n})} \]
Stephenson II Timed Curve

Kinematic synthesis equations solved for the first time

Found 1,529,788 roots
Over 3,563,520 trials
Computation lasted 24 hours
Using a laptop GPU

Estimated to find 99% of all finite roots

* Cognate structure reduced tracking requirements 50%
Application

Flat terrain

Typical motion

Running robot
Complex terrain

Greater strides would be useful

Longer flight phase
Design requirements for running:

- Cyclic motion
- Special mechanical advantage that pairs with an external spring
- Extra feature: Mech. adv. adjustability

![Diagram showing leg stroke, energy storage region, and foot point with energy released and stored areas.](image-url)
Leg Mechanism
Design Work Performed With This Result

Running robot

Weight (249 g)

Adjustment leads to higher powered behavior (High power mode)
Wrap Up

• Homotopy solvers (Bertini) allow design space exploration for mechanisms

• Stochastically generating startpoints with certain properties can save a lot on computation

• Finite Root Generation scales approximately linearly by the actual number of finite roots (essentially exploiting sparse monomial structures)

• Many six-bar design problems still unsolved (but they are being zeroed in on)
Thank you!